Lecture 8. Numerical Approximation: Euler's Method

In this section, we will talk about a numerical procedure for solving ordinary differential equations with a given initial value, which is called Euler method.

Let's look at an example.

Example 1

$$\boxed{\frac{dy}{dx} = y + 1, \qquad y(0) = 1} \tag{1}$$

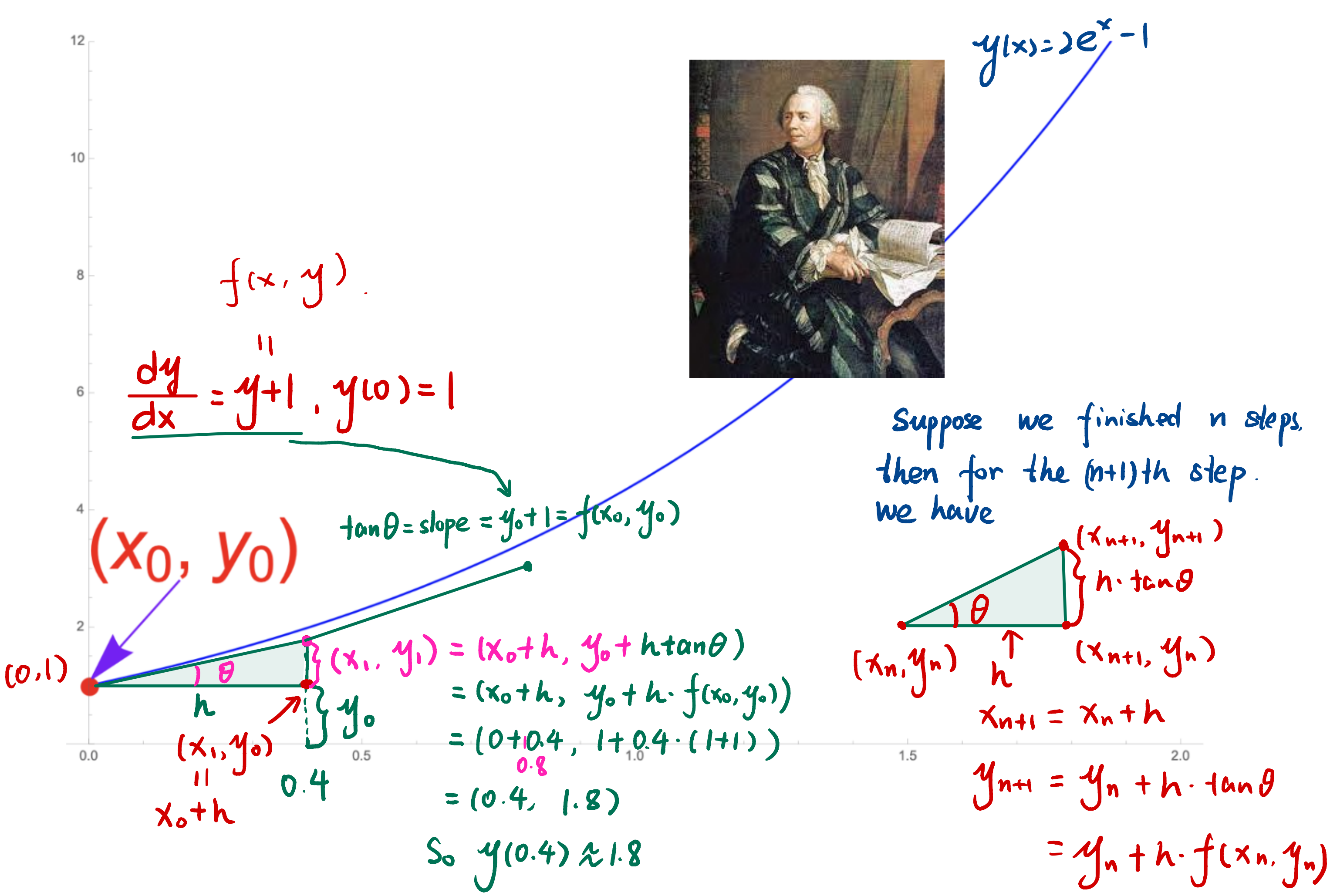
This is a seprable differential equation which we know how to solve. In practice, we will be given more complicated functions $f(x, y)$ in a more genal form:

$$\frac{dy}{dx} = f(x, y), \quad f(x_0) = y_0.$$

Using the method of solving separable equations (Section 1.4), we know the solution of Eq. (1) is $y(x) = 2e^x - 1$. I'll leave this as an exercise for you to check.

Now, let's "pretend" that we don't know how the solution curve look like.

How do we approximate the solution curve with the information given in Eq. (1)? 🤔



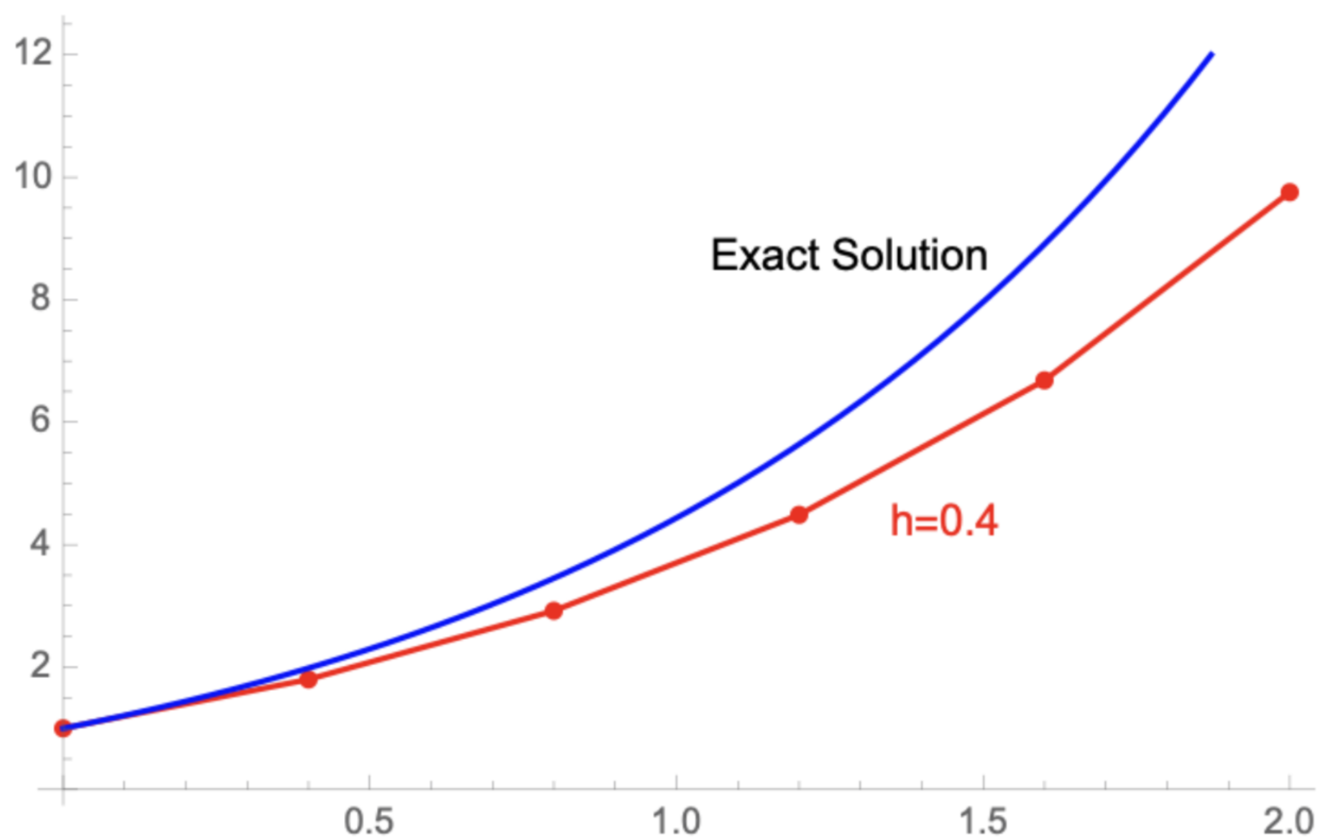
Let $h = 0.4$ (step size). We compute the following table:

$y(2) \approx ?$

Table 1

	$x_n = x_{n-1} + h$	$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$	(x_n, y_n)
$n = 0$	$x_0 = 0$	$y_0 = 1$	$(0, 1)$
$n = 1$	$x_1 = x_0 + h = 0.4$	$y_1 = y_0 + h \cdot (y_0 + 1) = 1 + 0.4 \cdot 2 = 1.8$	$(0.4, 1.8)$
$n = 2$	$x_2 = x_1 + h = 0.8$	$y_2 = y_1 + h \cdot (y_1 + 1) = 1.8 + 0.4 \cdot 2.8 = 2.92$	$(0.8, 2.92)$
$n = 3$	$x_3 = x_2 + h = 1.2$	$y_3 = y_2 + h \cdot (y_2 + 1) = 4.488$	$(1.2, 4.488)$
$n = 4$	$x_4 = x_3 + h = 1.6$	$y_4 = y_3 + h \cdot (y_3 + 1) = 6.6832$	$(1.6, 6.6832)$
$n = 5$	$x_5 = x_4 + h = 2$	$y_5 = y_4 + h \cdot (y_4 + 1) = 9.75648$	$(2, 9.75648)$

Then we plot those points on the xy-plane and connect them as those red straight lines.



How do we increase the accuracy of this approximation?

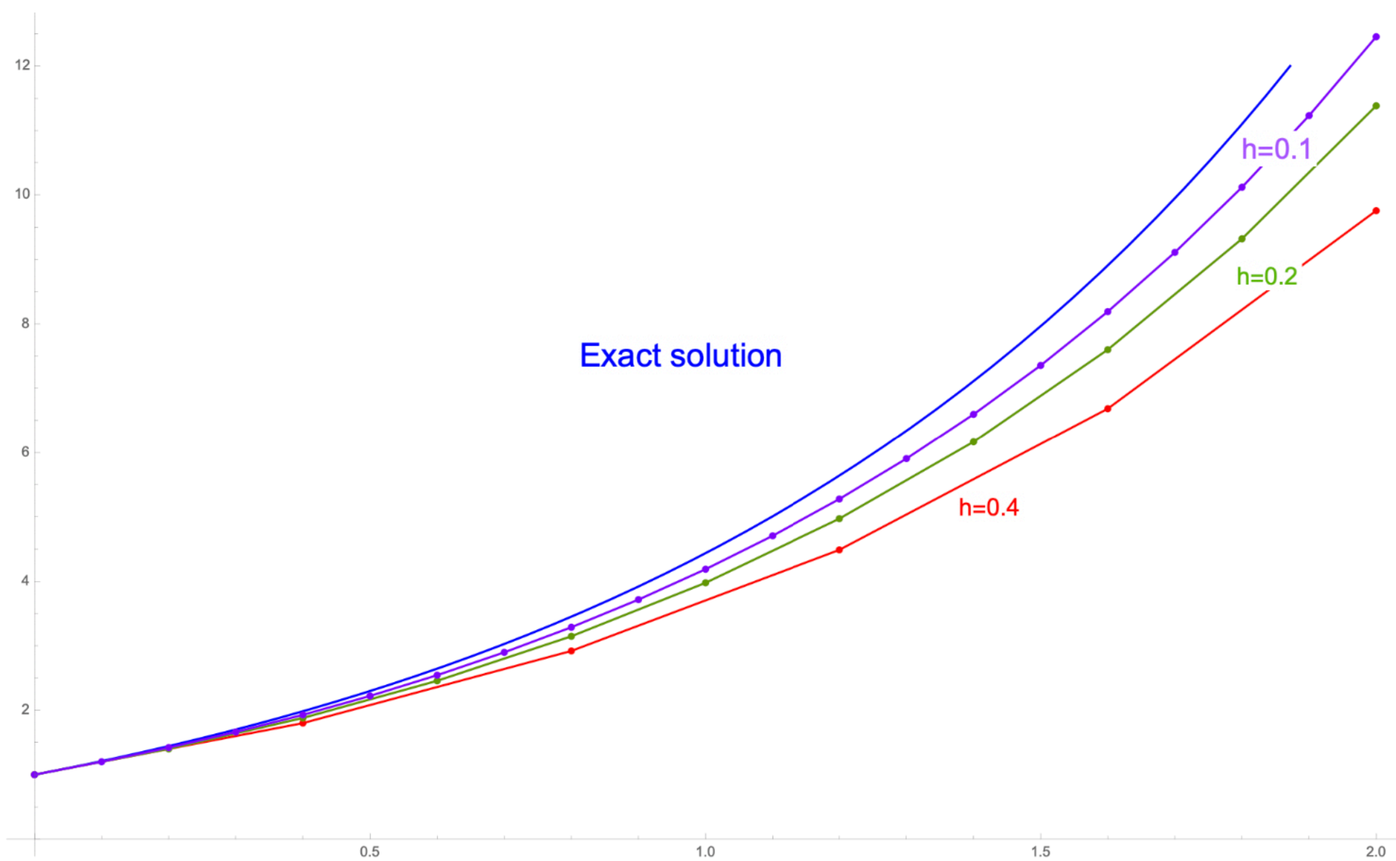
Let's try smaller step size $h = 0.2$, and compute the points similar to **Table 1**. We have the following points:

$h=0.2$

(0,1)
(0.2, 1.4)
(0.4, 1.88)
(0.6, 2.456)
(0.8, 3.1472)
(1, 3.97664)
(1.2, 4.971968)
(1.4, 6.1663616)
(1.6, 7.59963392)
(1.8, 9.319560704)
(2, 11.3834728448)

$h=0.1$

0	1
0.1	1.2
0.2	1.42
0.3	1.662
0.4	1.9282
0.5	2.22102
0.6	2.543122
0.7	2.8974342
0.8	3.28717762
0.9	3.715895382
1	4.1874849202
1.1	4.70623341222
1.2	5.276856753442
1.3	5.9045424287862
1.4	6.59499667166482
1.5	7.3544963388313
1.6	8.18994597271443
1.7	9.10894056998588
1.8	10.1198346269845
1.9	11.2318180896829
2	12.4549998986512



Algorithm The Euler Method

Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0,$$

Euler's method with step size h constists of applying the iterative formula $x_{n+1} = x_n + h$

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad (n \geq 0)$$

to calculate sucessive approximations $y_1, y_2, y_3, \cdots$ to the [true] values $y(x_1), y(x_2), y(x_3), \cdots$ of the [exact] solution $y = y(x)$ at the points $x_1, x_2, x_3, \cdots$, respectively.

Example 2 The exact solution $y(x)$ is given for the inital value problem. Apply Euler's method twice to approximate to this solution on the interval $[0, \frac{1}{2}]$, first with step size $h = 0.25$, then with step size $h = 0.1$. Compare the three-decimal-place values of the two approximations at $x = \frac{1}{2}$ with the value $y(\frac{1}{2})$ of the actual solution.

$$y' = \underbrace{-3x^2 y}_{f(x,y)}, \quad y(0) = 3; \qquad y(x) = 3e^{-x^3}$$

$h = 0.25 = \frac{1}{4}$

$n=0$	$x_0 = 0$	$y_0 = 3$
$n=1$	$x_1 = 0.25$	$y_1 = y_0 + h \cdot f(x_0, y_0) = 3 + 0.25 \cdot 0 = 3$
$n=2$	$x_2 = 0.5$	$y_2 = y_1 + h \cdot f(x_1, y_1) = 3 + 0.25 \times (-3 \cdot 0.25^2 \cdot 3) = 2.859375$

$y_{appr1}(\frac{1}{2}) = 2.859 \qquad y_{actual}(\frac{1}{2}) = 3 \cdot e^{-(\frac{1}{2})^3} \approx 2.6475$

$h = 0.1$

$n=0$	$x_0 = 0$	$y_0 = 3$
$n=1$	$x_1 = 0.1$	$y_1 = y_0 + h \cdot f(x_0, y_0) = 3 + 0.1 \times 0 = 3$
$n=2$	$x_2 = 0.2$	$y_2 = y_1 + h \cdot f(x_1, y_1) = 3 + 0.1 \times (-3 \times 0.1^2 \times 3) = 2.991$
$n=3$	$x_3 = 0.3$	$y_3 = y_2 + h \cdot f(x_2, y_2) = 2.955108$
$n=4$	$x_4 = 0.4$	$y_4 = y_3 + h \cdot f(x_3, y_3) = 2.875320$
$n=5$	$x_5 = 0.5$	$y_5 = y_4 + h \cdot f(x_4, y_4) = 2.73730$

$y_{appr2}(\frac{1}{2}) = 2.73730$

Exercise 3. Consider the differential equation $y' = -x - y$.
(1) Use Euler's method with $\Delta x = 0.1$ to estimate y when $x = 1.4$ for the solution curve satisfying $y(1) = 1$.
(2) Use Euler's method with $\Delta x = 0.1$ to estimate y when $x = 2.4$ for the solution curve satisfying $y(1) = 0$.

Solutions.
(1) Euler's method for $y' = -x - y$ with $y(1) = 1$ with $h = \Delta x = 0.1$ gives:

	$x_n = x_{n-1} + h$	$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$
$n = 0$	$x_0 = 1$	$y_0 = 1$
$n = 1$	$x_1 = x_0 + h = 1.1$	$y_1 = y_0 + h \cdot (-x_0 - y_0) = 1 + 0.1 \cdot (-2) = 0.8$
$n = 2$	$x_2 = x_1 + h = 1.2$	$y_2 = y_1 + h \cdot (-x_1 - y_1) = 0.8 + 0.1 \cdot (-1.1 - 0.8) = 0.61$
$n = 3$	$x_3 = x_2 + h = 1.3$	$y_3 = y_2 + h \cdot (-x_2 - y_2) = 0.61 + 0.1 \cdot (-1.2 - 0.61) = 0.429$
$n = 4$	$x_4 = x_3 + h = 1.4$	$y_4 = y_3 + h \cdot (-x_3 - y_3) = 0.429 + 0.1 \cdot (-1.3 - 0.429) = 0.2561$

So $y(1.4) \approx 0.2561$.

(2) Similarly, Euler's method for $y' = -x - y$ with $y(1) = 0$ with $h = \Delta x = 0.1$ gives:

	$x_n = x_{n-1} + h$	$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$
$n = 0$	$x_0 = 1$	$y_0 = 0$
$n = 1$	$x_1 = x_0 + h = 1.1$	$y_1 = y_0 + h \cdot (-x_0 - y_0) = 0 + 0.1 \cdot (-1) = -0.1$
$n = 2$	$x_2 = x_1 + h = 1.2$	$y_2 = y_1 + h \cdot (-x_1 - y_1) = -0.1 + 0.1 \cdot (-1.1 + 0.1) = -0.1 - 0.1 = -0.2$
$n = 3$	$x_3 = x_2 + h = 1.3$	$y_3 = y_2 + h \cdot (-x_2 - y_2) = -0.2 + 0.1 \cdot (-1.2 + 0.2) = -0.2 - 0.1 = -0.3$
$n = 4$	$x_4 = x_3 + h = 1.4$	$y_4 = y_3 + h \cdot (-x_3 - y_3) = -0.3 + 0.1 \cdot (-1.3 + 0.3) = -0.3 - 0.1 = -0.4$
...	...	...

We note that this is proceeding linearly, because the slope is constant (we always have $h \cdot f(x_{n-1}, y_{n-1}) = -0.1.$)

Thus we have $y(2.4) = -1.4$ ($= h \cdot (\text{steps needed to get to } 2.4) = 0.1 \times -14$)

In this case the Euler approximation is exact, because our initial condition happens to lie on a linear solution to the differential equation. In fact, the solution to the initial value problem $y' = -x - y$ with $y(1) = 0$ is $y(x) = 1 - x$ (try to solve it as a linear 1st order equation).